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ORIGINAL ARTICLE

A note on the solution of general Falkner-Skan problem by two novel semi-analytical techniques

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Abstract The aim of this paper is to give a presentation of two new iterative methods for solving non-linear differential equations, they are successive linearisation method and spectral homotopy perturbation method. We applied these techniques on the non-linear boundary value problems of Falkner-Skan type. The methods used to find a recursive former for higher order equations that are solved using the Chebyshev spectral method to find solutions that are accurate and converge rapidly to the full numerical solution. The methods are illustrated by progressively applying the technique to the Blasius boundary layer equation, the Falkner-Skan equation and finally, the magnetohydrodynamic (MHD) Falkner-Skan equation. The solutions are compared to other methods in the literature such as the homotopy analysis method and the spectral-homotopy analysis method with focus on the accuracy and convergence of this new techniques.

1. Introduction

Most problems that arise in engineering are nonlinear with no analytical solutions and developing new methods that give rapid convergence, are robust and easy to use is a core function of numerical analysis. In the last few decades several new iterative, perturbation and non-perturbation methods have been developed and used to solve nonlinear equations. Abbasbandy [1], Basto et. al. [2], Chun [3], Feng [4] among others have recently proposed and developed several iterative methods for solving nonlinear equations. One of the most successful methods for solving nonlinear equations and dating back to the early 1980’s is the Adomian decomposition method (ADM) [5,6] which uses a decomposition of the nonlinear operator as a series that in most instances gives fast convergence to the exact solution.

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The ADM solution however may not always guarantee convergence of the solution series. Some of the recently methods developed (see Refs. [1–3,7]) rely on attempts to improve the Newton-Raphson method by using the Adomian decomposition technique to improve the convergence of the solution. Such attempts, as in the case of Babolian and Biazar [7] have not always been entirely successful, see Basto et al. [2]. Indeed, even the algorithm developed by Basto et al. [2] only has third-order convergence. Another drawback of some of these techniques is that they use higher order differential derivatives. The two-step method proposed in Feng [4] aims to address this weakness. Nonetheless, even this method only gives quadratic convergence, equivalent to the Newton-Raphson method.

In addition to iterative methods is the homotopy perturbation method (HPM), it was proposed first by the Chinese researcher J.H. He in 1998 [8–11], he used this method to solve Lighthill equation [8], Duffing equation [9] and Blasius equation [12] and he has been also used this method to solve the nonlinear wave equations [11] and the boundary value problems [13,14], he has successfully been applied to solve many types of linear and nonlinear differential equations. The Homotopy perturbation method has been recently intensively studied by scientists and they used it for solving nonlinear problems in nonlinear mechanics, see for instance [11,15–18] and some modifications of this method have published [19–23] to facilitate and accurate the calculations and accelerate the rapid convergence of the series solution and reduce the size of work. However, as admitted by He himself [24], the method has many theoretical and application limitations. The basic premise of the homotopy perturbation method is that the solution of the nonlinear equation can always be written as a power series in terms of an embedding parameter p that monotonically decreases from unit to zero as the equation is continuously deformed from the nonlinear problem to an easy to solve linear equation. Another kind of iterative methods, new and highly effective non-perturbation techniques for solving highly nonlinear equations is the homotopy analysis method (HAM). It has been proposed by Shijun Liao [25–28]. The homotopy analysis methods leads to convergent series solutions of strongly nonlinear problems, independent of any small or large physical parameter associated with the problem [28]. The method has applied and developed in different types of differential equation by Abbasbandy et al. [29–32], Motsa et al. [33,34] and Makukula and Motsa [35]. The basic idea underpinning the use of the homotopy analysis method is the replacement of a nonlinear equation by a system of ordinary differential equations (ODEs) that can easily be solved with the help of symbolic computation software such as Maple or Mathematica. The solution of this system of ODEs is used to form a convergent series which, as proved in Refs. [26,28], is the solution of the original nonlinear equation. However, despite its many documented successes, the homotopy analysis method suffers from a number of deficiencies. The recent spectral modification of the HAM by Motsa et al. [36] attempts to address these identified limitations of the HAM by, for example, replacing the system of ordinary differential equations with algebraic equations and using the Chebyshev pseudo-spectral method to solve the higher order deformation equations. The method essentially uses Chebyshev polynomials as basis functions to speed up convergence of the method.

The first new method present in this paper that uses successive linearisation to construct recursive formulae for higher order equations, see Makukula et al. [37]. The method, hereinafter referred to as the successive linearisation method (SLM). The second method is the spectral homotopy perturbation method (SHPM), it is an alternative implementation of the homotopy perturbation method for nonlinear problems in bounded domains. We used the Chebyshev pseudo-spectral method to solve the higher order deformation equations. In the proposed spectral homotopy perturbation method, the auxiliary linear operator is defined in terms of the Chebyshev spectral collocation differentiation matrix described in Ref. [38]. The methods (SLM and SHPM) illustrated by progressively applying the techniques to the Blasius flat-plate boundary layer problem, the Falkner-Skan equation and finally, the magnetohydrodynamic (MHD) Falkner-Skan equation. The Falkner-Skan equations were solved recently using the homotopy analysis method, Liao [39], the homotopy perturbation method, Alizadeh-Pahlavan and Borjian-Boroujeni [40] and the spectral-homotopy analysis method, Motsa et al. [41]. The current solutions are compared to those in Refs. [39–41] with focus on accuracy, fast convergence and efficiency of these new techniques. The SLM and SHPM results are also compared with numerical results obtained using the very efficient MATLAB in-built bvp4c routine which is a boundary value solver based on the adaptive Lobatto quadrature scheme [42,43].

2. Problem statement

The equation for the two-dimensional flow of a viscous incompressible fluid past a semi-infinite surface with stream-wise pressure gradient is given by Ref. [44] as:

$$f'''(\eta)+\alpha f(\eta)f''(\eta)+\beta[1-f'(\eta)f'(\eta)]=0 \tag{1}$$

with boundary conditions

$$f(0)=\gamma, \quad f'(0)=\lambda, \quad f'(\infty)=1, \tag{2}$$

where f is the dimensionless stream function, α and β are positive parameters, γ is the wall mass transfer coefficient, λ is the wall stretching parameter and η is the dimensionless normal coordinate. The case $\alpha=\frac{1}{2}$ and $\beta=0$ gives the classical Blasius equation

$$f'''+\frac{1}{2}ff''=0. \tag{3}$$

When $\alpha=1$, Eq. (1) reduces to the Falkner-Skan equation

$$f'''+ff''+\beta[1-(f')^2]=0. \tag{4}$$

In the case of magnetohydrodynamics, it can easily be shown that the MHD Falkner-Skan equation is [2]

$$f''' + ff'' + \beta[1-(f')^2] - M^2(f'-1) = 0. \tag{5}$$

In general, the three Eqs. (3)–(5) take the form

$$f''' + c_1 ff'' + c_2(f')^2 + c_3 f' + c_4 = 0. \tag{6}$$

In this paper we investigate the solution of Eqs. (3)–(5) using a successive linearisation technique subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \tag{7}$$

3. Successive linearisation method approach

To investigate the successive linearisation application on Eq. (6) we assume that the solution $f(\eta)$ can be obtained as

$$f(\eta) = F_i(\eta) + \sum_{m=0}^{i-1} f_m(\eta), \quad i = 1,2,3,\ldots \tag{8}$$

where F_i are unknown functions and f_m, $(m \geq 1)$ are approximations which are obtained by recursively solving the linear part of the equation system that results from substituting Eq. (8) in the governing Eq. (6) gives

$$F_i''' + a_{1,i-1}F_i'' + a_{2,i-1}F_i' + a_{3,i-1}F_i + c_1 F_i'' F_i + c_2 F_i' F_i' = \phi_{i-1}, \tag{9}$$

where the coefficient parameters $a_{k,i-1}$, $(k = 1,\ldots,3)$ and ϕ_{i-1} are defined as

$$a_{1,i-1} = c_1 \sum_{m=0}^{i-1} f_m, \quad a_{2,i-1} = 2c_2 \sum_{m=0}^{i-1} f_m' + c_3,$$
$$a_{3,i-1} = c_1 \sum_{m=0}^{i-1} f_m'', \tag{10}$$

$$\phi_{i-1} = -\left[\sum_{m=0}^{i-1} f_m''' + c_1 \sum_{m=0}^{i-1} f_m'' \sum_{m=0}^{i-1} f_m + c_2\left(\sum_{m=0}^{i-1} f_m'\right)^2 + c_3 \sum_{m=0}^{i-1} f_m' + c_4\right] \tag{11}$$

Starting from the initial approximation given by

$$f_0(\eta) = \eta + e^{-\eta} - 1, \tag{12}$$

which is chosen to satisfy the boundary conditions (7), the subsequent solutions f_m $(m \geq 1)$ are obtained by successively solving the linearized form of Eq. (9) which is given as

$$f_i''' + a_{1,i-1}f_i'' + a_{2,i-1}f_i' + a_{3,i-1}f_i = \phi_{i-1}, \tag{13}$$

subject to the boundary conditions

$$f_i(0) = f_i'(0) = f_i'(\infty) = 0. \tag{14}$$

Once each solution for f_i, $(i \geq 1)$ has been found from iteratively solving Eq. (13), the approximate solution for $f(\eta)$ is obtained as

$$f(\eta) \approx \sum_{m=0}^{K} f_m(\eta), \tag{15}$$

where K is the order of SLM approximation. Eq. (15), is obtained by assuming that F_i becomes increasingly smaller as i becomes large, that is

$$\lim_{i\to\infty} F_i = 0. \tag{16}$$

Since the coefficients parameters and the right hand side of Eq. (13) for $i = 1,2,3,\ldots$, are known (from previous iterations), the equation can easily be solved using numerical methods such as finite differences, finite elements, Runge-Kutta based shooting methods or collocation methods. In this paper, Eq. (13) is solved using the Chebyshev spectral collocation method (see for example, [38,45,46]). The method is based on the Chebyshev polynomials defined on the interval [-1,1] by

$$T_k(\xi) = \cos[k\cos^{-1}(\xi)].$$

To implement the method, the physical region $[0,\infty)$ is transformed into the region [-1,1] using the domain truncation technique whereby the problem is solved in the interval $[0,L]$ instead of $[0,\infty)$. This leads to the mapping

$$\frac{\eta}{L} = \frac{\xi+1}{2} \quad -1 \leq \xi \leq 1$$

where L is the scaling parameter used to invoke the boundary condition at infinity. We use the popular Gauss-Lobatto collocation points [38,45,46]) to define the Chebyshev nodes in [-1,1], namely;

$$\xi_j = \cos\frac{\pi j}{N} \quad -1 \leq \xi \leq 1, \quad j = 0,1,2,\ldots,N.$$

The variable f_i is approximated by the interpolating polynomial in terms of its values at each of the collocation points by employing the truncated Chebyshev series of the form

$$f_i(\xi) = \sum_{k=0}^{N} f_i(\xi_k)T_k(\xi_j), \quad j = 0,1,\ldots,N. \tag{17}$$

where T_k is the k th Chebyshev polynomial. Derivatives of the variables at the collocation points may be represented by

$$\frac{d^r f_i}{d\eta^r} = \sum_{k=0}^{N} \boldsymbol{D}_{kj}^r f_i(\xi_k), \quad j = 0,1,\ldots,N$$

where r is the order of differentiation and $\boldsymbol{D} = \frac{2}{L}D$, with D being the Chebyshev spectral differentiation matrix (see, for example [45,46]) whose entries are defined as

$$\left.\begin{aligned} D_{jk} &= \frac{c_j}{c_k}\frac{(-1)^{j+k}}{\xi_j - \xi_k} \quad j \neq k; j,k = 0,1,\ldots,N, \\ D_{kk} &= -\frac{\xi_k}{2(1-\xi_k^2)} \quad k = 1,2,\ldots,N-1, \\ D_{00} &= \frac{2N^2+1}{6} = -D_{NN}. \end{aligned}\right\} \tag{18}$$

Substituting Eqs. (17) and (18) into Eqs. (13) and (14) gives

$$\boldsymbol{A}_{i-1}\boldsymbol{F}_i = \boldsymbol{\Phi}_{i-1},$$

subject to

$$f_i(\xi_N) = 0, \quad \sum_{k=0}^{N} \boldsymbol{D}_{Nk} f_i(\xi_k) = 0, \quad \sum_{k=0}^{N} \boldsymbol{D}_{0k} f_i(\xi_k) = 0, \tag{19}$$

where

$$\boldsymbol{A}_{i-1} = \boldsymbol{D}^3 + a_{1,i-1}\boldsymbol{D}^2 + a_{2,i-1}\boldsymbol{D} + a_{3,i-1},$$

$$\boldsymbol{F}_i = \left[f_i(\xi_0), f_i(\xi_1), \ldots, f_i(\xi_1)\right]^{\mathrm{T}},$$

$$\boldsymbol{\Phi}_{i-1} = \left[\varphi_{i-1}(\xi_0), \varphi_{i-1}(\xi_1), \ldots, \varphi_{i-1}(\xi_N)\right]^{\mathrm{T}},$$

In the above definitions T stands for transpose and $a_{k,i-1}(k=1,2,3)$ denotes $\boldsymbol{a}$ diagonal matrix of size $(N+1)\times(N+1)$. The boundary condition $f_i(\xi_N)=0$ is implemented by deleting last row and last column of $\boldsymbol{A}_{i-1}$, and deleting the last rows of $\boldsymbol{F}_i$ and $\boldsymbol{\Phi}_{i-1}$. The derivative boundary conditions in Eq. (19) are then imposed on the resulting first row and last row of $\boldsymbol{A}_{i-1}$ and setting the first and last rows of $\boldsymbol{F}_i$ and $\boldsymbol{\Phi}_{i-1}$ to be zero. The solutions for $f_i(\xi_1) f_i(\xi_2), \ldots, f_i(\xi_{N-1})$ are then obtained iteratively from solving

$$\boldsymbol{F}_i = \boldsymbol{A}_{i-1}^{-1}\boldsymbol{\Phi}_{i-1}.$$

4. Spectral-homotopy perturbation method approach

To describe the spectral-homotopy perturbation method, we consider the following general second order boundary value problem:

$$u''(x) + a(x)u'(x) + b(x)u(x) + N[u,x] = F(x) \tag{20}$$

subject to the boundary conditions:

$$u(-1) = u(1) = 0$$

where $x \in [-1,1]$, $a(x)$, $b(x)$ and $F(x)$ are known continuous functions and N is nonlinear function of x and u. The differential Eq. (20) can be written in the following operator form:

$$L[u(x)] + N(x,u) = F(x)$$

where

$$L = \frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x)$$

we construct the following homotopy:

$$\begin{aligned} H(U,p) = L[U] - L[u_0] + pL[u_0] \\ + p[N(U,x) - F(x)] = 0, \end{aligned} \tag{21}$$

where u_0 is an initial solution of the nonhomogeneous linear part of governing differential Eq. (20) given by:

$$u_0'' + a(x)u_0' + b(x)u_0 = F(x) \tag{22}$$

subject to the boundary conditions:

$$u_0(-1) = u_0(1) = 0.$$

If an exact solution of Eq. (22) can be found, we can also use this method (SHPM) to find approximate solution $u_0(x)$. Obviously, from Eq. (21) we have

$$H(U,0) = L[U] - L[u_0] = 0,$$

$$H(U,1) = L[U] + [N(U,x) - F(x)] = 0.$$

assume that the solution of Eq. (21) can be written as a power series in p:

$$U = u_0 + pu_1 + p^2u_2 + \ldots \tag{23}$$

substitute Eq. (23) in Eq. (21) and comparing the coefficientss of p^i starting form p^1 : to get the following system differential equation:

$$u_1'' + a(x_j)u_1' + b(x_j)u_1 = \Phi_1 \tag{24}$$

subject to the boundary conditions:

$$u_1(-1) = u_1(1) = 0. \tag{25}$$

where

$$\Phi_1 = -\left[u_0'' + a(x_j)u_0' + b(x_j)u_0 + N(U,x_j) - F(x_j)\right].$$

and x_j are the Chebyshev nodes ([8,10,35]) defined in $[-1,1]$ by

$$x_j = \cos\frac{j\pi}{N}, \quad j = 0,1,2,\ldots,N \tag{26}$$

The derivatives of the functions $u_i(x_j), j = 1,2,3,\ldots,N$ at the collocation points x_j are represented as

$$\frac{d^r u_i(x_j)}{dx^r} = \sum_{k=0}^{N} D_{kj}^r u_i(x_j) \tag{27}$$

where D is the Chebyshev spectral differentiation matrix whose defined by Eq. (18). Substituting Eqs. (27) and (26) in Eq. (24) yields a system of equations of the form

$$\boldsymbol{A}u_1 = \Phi_1 \tag{28}$$

where

$$\boldsymbol{A} = D^2 + \operatorname{diag}[a(x_j)]D + \operatorname{diag}[b(x_j)]$$

where diag [ ] is a diagonal matrix of size $(N+1)\times(N+1)$. The matrix $\boldsymbol{A}$ has dimensions $(N+1)\times(N+1)$ while matrices u_1 and Φ_1 have a dimensions $(N+1)\times 1$. To implement the boundary conditions of Eq. (25) we delete the first and the last rows and columns of $\boldsymbol{A}$, the first and last elements of Φ_1 and u_1 are also deleted. This reduces the dimension of $\boldsymbol{A}$ to $(N-1)\times(N-1)$ and those of u_1 and Φ_1 to $(N-1)\times 1$.

The solution to Eq. (28) is

$$u_1 = \boldsymbol{A}^{-1}\Phi_1. \tag{29}$$

Eq. (29) gives the second approximation of the differential Eq. (20). To evaluate the approximations $u_i, i = 2,3,4,\ldots$ we can make a comparison between the coefficients of p^i of both sides in Eq. (21) to get more accuracy for the solution of Eq. (20) by usin the form

$u_i = \boldsymbol{A}^{-1}\boldsymbol{\Phi}_i,$

now substitute u_is in Eq. (23) and setting $p=1$ to obtain an approximate solution of Eq. (20).

To apply the spectral homotopy perturbation method on Eq. (6), first use the domain truncation method to approximate the domain of the problem from $[0,\infty)$ to $[0,L]$ where L is chosen to be sufficiently large. We then transform $[0,L]$ to the domain $[-1,1]$ on which the Chebyshev spectral method can be applied by using the transformations

$$x = \frac{2\eta}{L} - 1 \tag{30}$$

where $x \in [-1,1]$. It is also convenient to make the boundary conditions homogeneous by making use of the transformation

$$f(\eta) = u(x) + f_0(\eta) \tag{31}$$

where

$$f_0(\eta) = \frac{1}{e^{-\lambda L} - 1}\left[\frac{1}{\lambda} - \eta - \frac{1}{\lambda}e^{-\lambda\eta}\right]$$

and λ is a spatial scaling parameter. Substituting Eq. (30) and Eq. (31) in the governing Eq. (6) gives

$$a_3u'''(x) + a_2u''(x) + a_1u'(x) + a_0u(x) + \frac{4}{L^2}c_1u(x)u''(x) + \frac{4}{L^2}c_2(u'(x))^2 = \Phi(\eta) \tag{32}$$

subject to the boundary conditions

$$u(-1) = u'(-1) = u'(1) = 0$$

where

$$a_0 = c_1f_0'', \quad a_1 = \tfrac{2}{L}\left(2c_2f_0' + c_3\right),$$
$$a_2 = \tfrac{4}{L^2}c_1f_0, \quad a_3 = \tfrac{8}{L^3}$$

and

$$\Phi(\eta) = -\left(f_0''' + c_1f_0f_0'' + c_2(f_0')^2 + c_3f_0' + c_4\right)$$

now we may choose the following linear differential operator for Eq. (32)

$$L = a_3\frac{d^3}{dx^3} + a_2\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0 \tag{33}$$

The initial approximation for the solution of Eq. (32) is obtained from the solution to the nonhomogeneous linear part of Eq. (32) which is

$$a_3u_0'''(x) + a_2u_0''(x) + a_1u_0'(x) + a_0u(x) = \Phi_0(\eta) \tag{34}$$

subject to the boundary conditions

$$u_0(-1) = u_0'(-1) = u_0'(1) = 0$$

where u_0 is initial approximation for the solution of Eq. (32) and $\Phi_0(\eta) = \Phi(\eta)$. If an exact solution of Eq. (34) cannot be found, we may use SHPM to find it. We now construct the homotopy:

$$\begin{aligned} H(U;p) &= L[U] - L[u_0] + pL[u_0] \\ &\quad + p(N(U,x) - \Phi) = 0 \end{aligned} \tag{35}$$

where where $p \in [0,1]$ is an embedding parameter and U is an approximate series solution of Eq. (32) given by

$$U = u_0 + pu_1 + p^2u_2 + \cdots. \tag{36}$$

and $N(U,x)$ is the nonlinear part of Eq. (32). Substitute Eq. (33) in Eq. (35) and compare between the power of p^1 to obtain the following equation

$$\boldsymbol{A}u_1 = \boldsymbol{\Phi}_1 \tag{37}$$

subject to the boundary conditions

$$u_1(-1) = u_1'(-1) = u_1'(1) = 0 \tag{38}$$

where

$$\begin{aligned} \boldsymbol{A} &= a_3D^3 + \text{ diag }[a_2(\eta_i)]D^2 \\ &\quad + \text{diag }[a_1(\eta_i)]D + \text{ diag }[a_0(\eta_i)] \end{aligned} \tag{39}$$

$$\boldsymbol{\Phi}_1 = -\left[\boldsymbol{A}u_0 + \frac{4}{L^2}c_1u_0(D^2u_0'') + \frac{4}{L^2}c_2(Du_0')^2 - \Phi_0\right]^{\mathrm{T}}$$

here T denotes transpose, diag[ ] is a diagonal matrix of size $(N+1)\times 1$, the solution of Eq. (37) can be given by

$$u_1 = \boldsymbol{A}^{-1}\boldsymbol{\Phi}_1$$

To get more higher order approximations for Eq. (32), we compare between the coefficients of $p^i, (i = 2,3,4,\ldots)$ in Eq. (35) to obtained the following approximations

$$u_i = \boldsymbol{A}^{-1}\boldsymbol{\Phi}_i \tag{40}$$

Table 1 Comparison of the SLM and SHPM ($\lambda = 0.58$) results against the HAM results of Refs. [20,31] and the SHAM [28] for $f''(0)$ in the Blasius flow.

SLM		SHPM		Motsa and Sibanda [28]		Pahlavan [31]		Liao [20]	
Order	$f''(0)$	Order	$f''(0)$	Order	$f''(0)$	Order	$f''(0)$	Order	$f''(0)$
1	0.361245	4	0.332059	2	0.332253	2	0.298022	5	0.256390
2	0.332939	5	0.332058	4	0.332062	4	0.327531	10	0.327756
3	0.332059	6	0.332057	6	0.332057	6	0.330855	20	0.331851
4	0.332057	7	0.332057	8	0.332057	8	0.331503	30	0.332040
5	0.332057	8	0.332057	10	0.332057	10	0.331807	40	0.332055

Table 2 Comparison between the SLM, SHPM ($\lambda = 0.58$) results and the *bvp4c* numerical results for $f'(\eta)$ at selected values of η for the Blasius boundary layer.

	SLM			Numerical	SHPM		
η	2nd order	3rd order	4th order		5th order	4th order	3rd order
0.40	0.133117	0.132765	0.132764	0.132764	0.132764	0.132764	0.132765
0.80	0.265417	0.264710	0.264709	0.264709	0.264709	0.264709	0.264711
1.20	0.394847	0.393778	0.393776	0.393776	0.393776	0.393776	0.393778
2.00	0.631592	0.629768	0.629766	0.629766	0.629766	0.629766	0.629770
4.00	0.957066	0.955522	0.955518	0.955518	0.955518	0.955516	0.955522
5.00	0.992065	0.991544	0.991542	0.991542	0.991542	0.991542	0.991542
6.00	0.999063	0.998973	0.998973	0.998973	0.998973	0.998973	0.998973
7.00	0.999928	0.999922	0.999922	0.999922	0.999922	0.999922	0.999921
8.00	0.999996	0.999996	0.999996	0.999996	0.999996	0.999996	0.999996
9.00	1	1	1	1	1	1	1

Table 3 Comparison between the SLM, SHPM ($\lambda = 1.5$) results and the *bvp4c* numerical results for $f''(0)$ at selected values of β for the Falkner-Skan.

β	SLM			Numerical	SHPM		
	1st order	2nd order	3rd order		5th order	4th order	3rd order
0.4	0.855900	0.854422	0.854421	0.854421	0.854421	0.854422	0.854423
0.8	1.126462	1.120286	1.120268	1.120268	1.120268	1.120268	1.120268
1.2	1.352360	1.335807	1.335721	1.335721	1.335721	1.335721	1.335721
1.6	1.551938	1.521742	1.521514	1.521514	1.521514	1.521514	1.521514
2.0	1.733718	1.687679	1.687218	1.687218	1.687218	1.687218	1.687219

Table 4 Comparison between the present successive linearisation results and the 10th order Homotopy-Padé results of Ref. [2] for $m = 2$, and the *bvp4c* numerical results for the MHD Falkner-Skan.

M	1st order	2nd order	3rd order	4th order	Numerical	Ref. [2]
1	1.74769149	1.71959955	1.71946569	1.71946568	1.71946565	1.71947219
2	2.47595858	2.43959024	2.43949896	2.43949896	2.43949892	2.43949870
5	5.22633691	5.19097088	5.19095980	5.19095980	5.19095980	5.19095980
10	10.12176679	10.09677658	10.09677575	10.09677575	10.09677575	10.09677575
50	50.02613722	50.01944084	50.01944084	50.01944084	50.01944084	50.01944084

subject to the boundary conditions

$$u_i(-1) = u_i'(-1) = u_i'(1) = 0 \tag{41}$$

where

$$\Phi_i = -\left(\frac{4}{L^2}c_1\left[\sum_{n=0}^{i-1} f_n\left(D^2 f_{i-1-n}\right)\right] + \frac{4}{L^2}c_2\left[\sum_{n=0}^{i-1}\left(Df_n\right)\left(Df_{i-1-n}\right)\right]\right), \quad i \geq 2$$

The matrix $\boldsymbol{A}$ has dimensions $(N+1)\times(N+1)$ while matrices $\boldsymbol{\Phi}_i$ and u_i have a dimensions $(N+1)\times 1$. To implement the boundary conditions Eqs. (38) and (41) to the systems Eqs. (37) and (40) respectively, we delete the last row and last column of $\boldsymbol{A}$ and delete the last rows of $u_1, u_i, \boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_i$ also we replace the resulting of first and last rows of the modified matrix $\boldsymbol{A}$ and setting the resulting of first and last rows of the modified matrices $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_i$ to be zero. then the solution of Eq. (32) is given by substitute the series u_i in Eq. (36) after setting $p = 1$.

5. Results and discussion

In this section we present the results showing the velocity distribution $f'(\eta)$ and the skin friction $f''(0)$ for the Blasius, the Falkner-Skan and the MHD Falkner-Skan equations. To assess the accuracy of the SLM and SHPM, comparison is made with numerical solutions obtained using the

Table 5 Comparison between the present SHPM results ($N=200$) and the 10th order Homotopy-Padé results of Ref. [2] for $m=2$, and the *bvp4c* numerical results for the MHD Falkner-Skan.

M	λ	1st order	2nd order	3rd order	4th order	Numerical	Ref. [2]
1	2.1	1.71946599	1.71946566	1.71946565	1.71946565	1.71946565	1.71947219
2	2.1	2.43949986	2.43949889	2.43949892	2.43949892	2.43949892	2.43949870
5	5.2	5.19095980	5.19095980	5.19095980	5.19095980	5.19095980	5.19095980
10	10.1	10.09677575	10.09677575	10.09677575	10.09677575	10.09677575	10.09677575
50	50.0	50.01944084	50.01944084	50.01944084	50.01944084	50.01944084	50.01944084

MATLAB *bvp4c* routine. Comparison is also made with results reported in other related studies. In generating the results presented in this study, $L=20$ and $N=100$ collocation points were used in the SLM and SHPM procedures.

Table 1 gives a comparison of the current results for $f''(0)$ with those of Liao [39], Alizadeh-Pahlavan and Borjian-Boroujeni [40] and Motsa and Sibanda [41] at different orders of approximations of the solution series of the respective methods used. In their work, Alizadeh-Pahlavan and Borjian-Boroujeni [40], solved the Blasius problem by introducing an additional auxiliary parameter and suggested a straightforward approach for finding the best values of the HAM auxiliary parameter which plays a prominent in making the solution more convergent. Here we note that the numerical solution obtained by Howarth [47] is $f''(0)=0.33206$ while the numerical result by the Matlab *bvp4c* routine is $f''(0)=0.33205734$. In general all the methods give reasonably accurate results. It is however the computational efficiency of the method that is of particular interest in this study. The exact numerical result of $f''(0)=0.33205734$ was obtained at the 4th order of the solution series using the successive linearisation method (SLM) and at the 8th order in case of SHPM. The result $f''(0)=0.33206$ was obtained at the 3rd order. It is however instructive to note that in Liao [39] the homotopy analysis method gave $f''(0)=0.332055$ after forty iterations while four iterations of the SHAM in Motsa and Sibanda [41] gave $f''(0)=0.332062$. This indicates to us that the SLM and SHPM are more efficients than the HAM and other approaches in that the solution rapidly converges to the numerical results reported in Ref. [48] and to the *bvp4c* results.

Table 2 shows a comparison of the SLM, SHPM and the numerical values of $f'(\eta)$ at different values of η and different orders of the SLM and SHPM solution series of the Blasius boundary layer equation. Convergence of the SLM results to the numerical approximation is achieved at the fourth order and at fifth order of the SHPM. Accuracy however improves to the third and second orders as η increases.

In Table 3 the SLM and SHPM Falkner-Skan results for $f''(0)$ for different values of β are compared with the full numerical results. The results show that convergence of the two sets of results is achieved at the third-order of the SLM solution series for all values of β.

Tables 4 and 5 give three-way comparison of the SLM and SHPM respectively against the numerical and the 10th order Homotopy-Padé results of Abbasbandy and Hayat [47] for the MHD Falkner-Skan equation with increasing Hartman numbers. The accuracy and efficiency of the SLM generally improves with increasing Hartman numbers from about the fifth order when $M=1$ to the second order when $M=50$. We denote that from Table 5 the ccurency and efficincy of SHPM start from 3rd order for 8 decimal places.

6. Conclusion

In this paper, two new techniques for solving strongly nonlinear boundary value problems were proposed. We have shown through the solution of the Falkner-Skan type equations and comparison with other techniques that the successive linearisation method and spectral homotopy perturbation method are very accurate and converges very rapidly.

Some of the advantages of these methods are that solutions are obtained iteratively by integrating linear differential equations with no need of incorporating additional linearisation methods such as Newton's method as is normally done in other numerical approaches such as the Keller-Box and Runge-Kutta shooting methods. The SLM and SHPM are also global methods in that the solution is computed in the entire domain of the problem at every iteration step. Another desirable feature of the methods is that it is computationally efficient in that it requires only a few iterations to converge to the exact numerical results. Also The SLM and SHPM are simples and easy to use for solving the nonlinear problems and useful for finding an accurate approximation of the exact solution because the obtained governing equations, are presented in form of algebraic equations corresponding to our system of ordinary differential equations and the system of these algebraic equations is easy to solve. The methods are also very simple and straightforward to implement. Because of their efficiency, the methods has great potential to be used in solving non-linear equations in place of traditional numerical methods such as finite differences, shooting methods and pure collocation methods.

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